ISRAEL JOURNAL OF MATHEMATICS, Vol. 65, No. 2, 1989

A CONSISTENT COUNTEREXAMPLE IN THE THEORY OF COLLECTIONWISE HAUSDORFF SPACES†

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ABSTRACT
It is shown to be consistent that there is a normal first countable locally countable space which is not collectionwise Hausdorff and in which there is a closed discrete non-G_δ set which provides the counterexample to collectionwise Hausdorffness. This answers a question of P. Nyikos.

This paper will answer a question of P. Nyikos concerning collectionwise Hausdorff spaces. A space is collectionwise Hausdorff if and only if every closed discrete collection of points can be separated; in other words, it is possible to simultaneously assign a neighbourhood to each point so that no two neighbourhoods meet. In all known normal, first countable, not collectionwise Hausdorff spaces the closed discrete set witnessing the failure of collectionwise Hausdorffness is a G_δ. Hence Nyikos asked in [N] if it is possible to find a counterexample where the closed discrete set is not a G_δ. That is the purpose of this paper.

I thank J. Steprans for informing me of the problem and writing up the paper, and M. Goldstern for corrections and proofreading.

Let $\Omega = \{\alpha \in \omega_1; \alpha \text{ is a limit}\}$. If $S \subseteq \Omega$ is a stationary set, then a ladder system on S is a function $L : S \times \omega \to \omega_1$ satisfying the following conditions:

(1) $L(\alpha, n) < \alpha$,
(2) $L(\alpha, n) < L(\alpha, n + 1)$,
(3) $\bigcup\{L(\alpha, n); n \in \omega\} = \alpha$.

† Publication 349, partially supported by the BSF.
Received May 9, 1988

A function $g : \omega_1 \to 2$ is said to uniformize a function $c : S \to 2$ on a ladder system L if $|\{n \in \omega; g(L(\alpha, n)) \neq c(\alpha)\}| < \aleph_0$ for every $\alpha \in S$. A space can be constructed (as in [S2]) from a ladder system considering two copies of ω_1. By making the points of one copy, $\omega_1 \times \{0\}$, isolated and making the αth ladder of isolated points converge to α in the other copy, $\omega_1 \times \{1\}$, one obtains a space which is normal if and only if every function on the ladder system can be uniformized. The closed set of isolated points, $S \times \{1\}$, is G_δ if and only if the following property (*) fails:

$$(*) \qquad \text{If } f : \omega_1 \to \omega, \text{ then there is some } \alpha \in S \text{ and } k \in \omega \text{ such that } |\{n \in \omega; f(L(\alpha, n)) = k\}| = \aleph_0.$$

So it is enough to prove

THEOREM. *Suppose V satisfies* CH. *Then there is a forcing notion* **P** *which does not add reals, collapse cardinals, change cofinalities or the value of the exponential function such that in $V^{\mathbf{P}}$ there is a ladder system on $S \subseteq \Omega$, S stationary, which satisfies (*) and such that every colouring of the ladder system can be uniformized.*

PROOF. For notational simplicity assume $\aleph_2 = 2^{\omega_1}$.† The partial order $\mathbf{Q}(L, c)$ from [S] is designed to produce a function g which uniformizes c on L. The elements of $\mathbf{Q}(L, c)$ are functions $p : (\gamma + 1) \cap S \to 2$ where $\gamma \in \omega_1$ and

$$|\{n \in \omega; p(L(\alpha, N)) \neq c(\alpha)\}| < \aleph_0 \qquad \text{for every } \alpha \in \gamma + 1.$$

The ordering on $\mathbf{Q}(L, c)$ is set containment. Let **L** be the partial order for adding a ladder system generically with countable conditions (so the elements of **L** are countable functions $p : \eta \times \omega \to \eta$ and the ordering is again set containment — in fact **L** is simply adding a subset of ω_1 with countable conditions). Let the name for the generic function from $\omega_1 \times \omega$ to ω_1 be L. Note that L is not necessarily a ladder system since conditions (1), (2) and (3) need not necessarily be satisfied at all $\alpha \in \Omega$. Let S be a name for $\{\alpha \in \Omega; (1), (2)$ and (3) are satisfied$\}$. It follows that $L \restriction S$ will be a ladder system.

Define an iteration $\mathbf{P}(\omega_2)$ by induction as follows. Let $\mathbf{P}(1) = \mathbf{L}$. If $\mathbf{P}(\alpha)$ has been defined and $\underset{\sim}{c}_\alpha$ is a $V^{\mathbf{P}(\alpha)}$ name for a function from Ω to 2, then $\mathbf{P}(\alpha + 1) = \mathbf{P}(\alpha) * \mathbf{Q}(L \restriction S, \underset{\sim}{c}_\alpha)$. The iteration will have countable support. To

† Otherwise we can use a longer iteration, $\aleph_2$-cc will be preserved, by [S3] Ch. VIII, §2.

see that ω_1 is not collapsed and S is still stationary, see [S2].† This fact, together with the appropriate cardinal arithmetic in the ground model, makes it is possible to construct the iteration so that every name for a function $c: S \to 2$ appears and hence is uniformized. Let $\underset{\sim}{d}_\eta$ be a name for the generic colouring obtained from the ηth partial order.

Before continuing with the proof let us make a simple observation. Let D be the set of conditions, p, in $\mathbf{P}(\omega_2)$ for which there exists an ordinal $h(p)$, the "height" of p, such that for every γ

$$p(\gamma) \text{ is an actual function in } V \text{ (rather than a } P_\gamma\text{-name), and}$$
$$\text{domain}(p(\gamma)) = \rho(p) \text{ or domain}(p(\gamma)) = 0.$$

In [S2] it is shown that D is dense in $\mathbf{P}(\omega_2)$ and that any descending sequence of conditions from D has a lower bound provided the heights of the conditions converge to an ordinal without a ladder attached to it. In our case the ladder system is constructed generically and it will be possible to ensure that a ladder is not attached to a point by having (2) fail at that ordinal.

The remainder of this paper is devoted to proving that if G is $P(\omega_2)$ generic over V then (∗) holds in $V[G]$. To see this suppose that $p \in \mathbf{P}(\omega_2)$ and that it forces that $\underset{\sim}{f}$ is a counterexample. Let $\mathfrak{M}$ be a countable elementary submodel of $H(\omega_3)$ containing p, $\langle \underset{\sim}{\varsigma}_\alpha; \alpha < \omega_2 \rangle$, $\mathbf{P}(\omega_2)$ and $\underset{\sim}{f}$. Let $\delta = \mathfrak{M} \cap \omega_1$ and $\Sigma = \mathfrak{M} \cap \omega_2$. Choose an increasing sequence $\{\delta_n; n \in \omega\}$ cofinal in δ and let $\Sigma \setminus \{0\}$ be enumerated by the sequence $\{\sigma(n); n \in \omega\}$. Let $T = \bigcup\{{}^n2; n \in \omega\}$ and define $t^\wedge i = t \cup \{(\text{domain}(t), i)\}$ for $t \in T$. Let, for $s, t \in T, j \in \omega$, $s \underset{j}{\sim} t$ iff

$$t \restriction \{i \in \omega : \sigma(i) < \sigma(j)\} = s \restriction \{i \in \omega : \sigma(i) < \sigma(j)\}.$$

Induction on $m \in \omega$ will be used to construct, in $\mathfrak{M}$, $p_t \in \mathbf{P}(\omega_2)$, $k_t \in \omega$ and $\zeta_t \in \delta$ for each $t \in {}^m2$ satisfying conditions (4) to (11) below.

(4) $p_{t^\wedge i} \leqq p_t \in D$ for $i \in 2$,

(5) $\zeta_t \geqq \delta_{\text{domain}(t)}$,

(6) $(p_t(\sigma(j)))(\zeta_s) = t(j)$ if $|t| \geqq |s| > j$ by (4), it is enough to ensure this for $|t| = |s|$,

(7) either (a) $p_t \Vdash$ "$|\{\zeta \in \omega_1; \underset{\sim}{d}_{\sigma(j)}(\zeta) = t(j)$ for $j \in \text{domain}(t)\}| \leqq \omega$" or (b) $p_t \Vdash$ "$|\{\zeta \in \omega_1; \underset{\sim}{d}_{\sigma(j)}(\zeta) = t(j)$ for $j \in \text{domain}(t)$ and $\underset{\sim}{f}(\zeta) = k_t\}| = \omega_1$" and, moreover, k_t is the least such integer,

† Alternatively use [S3] Ch. III to show $\mathbf{P}(\omega_2)$ is proper, by Ch. V no ω-sequence of ordinals is added, by Ch. III $\mathbf{P}(\omega_2)$ satisfies the $\aleph_2$-cc and has (a dense subset of) power $\aleph_2$. So cardinal arithmetic is as required.

(8) if case (7b) holds then $p_t \Vdash$ "$\underset{\sim}{f}(\zeta_t) = k_t$",

(9) if $t \underset{j}{\sim} s$ then $p_t \restriction \sigma(j)$ and $p_s \restriction \sigma(j)$ are comparable,

(10) $p_\varnothing \leqq p$,

(11) $p_t \restriction \sigma(j) \Vdash$ "domain($p_t(\sigma(j))) \supset \delta_{\text{domain}(t)}$" for $j \in \text{domain}(t)$.

To show that the induction can be carried out suppose that p_t has been defined for $t \in {}^m2$. Let $\{t_i; i \in 2^{m+1}\}$ enumerate ${}^{m+1}2$. We define by induction on $n \leqq 2^m$ the following: $\langle p^n_{t_i} : i < {}^{m+1}2\rangle$, $\langle \zeta_{t_i} : i < n\rangle$ and $\langle k_{t_i} : i < n\rangle$ such that (for our $p^n_{t_i}, \zeta_{t_i}, k_{t_i}$) conditions (4), (7), (9), (11) hold, and condition (5) holds for $t \in \{t_i : i < n\}$, condition (6) holds for $s \in \{t_i : i < n\}$, $t \in {}^{m+1}2$ and condition (8) holds for $t \in \{t_i : i < n\}$.

For $n + 0$, we define $p^0_{t_i}$ by induction on $i < 2^{m+1}$ such that:

(i) $p^0_{t_i} \leqq p_{t_i \restriction m}$,

(ii) for every $l \leqq m+1$ and $s \in {}^m2$, if $t_i \restriction m \sim_l s$ then $p^0_{t_i} \restriction \sigma(l) \leqq p_s \restriction \sigma(l)$,

(iii) if $s \in \{t_j : j < i\}$ and $l \leqq m+1$ and $t_i \sim_l s$ then $p^0_{t_i} \restriction \sigma(l) \leqq p_s \restriction \sigma(l)$,

(iv) $p^0_{t_i}$ satisfies condition (7).

There are no problems — first by the induction hypothesis and condition (9) we can find $q^0_{t_i}$ satisfying (i), (ii), (iii), and then (iv) is satisfied by a dense set of members of the forcing notion.

For $n+1$, by condition (9) we can find $q \leqq p^n_{t_n \restriction m}$ such that:

$$\text{if } s \in {}^{m+1}2,\ l \leqq m+1,\ s \sim_l t_j \text{ then } q \restriction s(l) \leqq p^j_s \restriction s(l).$$

If possible, find $q_0 \leqq q$ such that q_0 forces the first alternative in (7). If this is not possible then

$$q \Vdash \text{"} | \{ \zeta \in \omega_1;\ \underset{\sim}{d}_{\sigma(j)}(\zeta) = (t_a{}^\frown 0)(j) \text{ for } j \in \text{domain}(t_n)\} | = \omega_1\text{"}.$$

Hence $q \Vdash$ "there is a least integer k such that

$$\underset{\sim}{A}\ | \{ \zeta \in \omega_1;\ \underset{\sim}{d}_{\sigma(j)}(\zeta) = (t_n)(j) \text{ for } j \in \text{domain}(t_a{}^\frown 0) \text{ and } \underset{\sim}{f}(\zeta) = k_{t_n}\} | = \omega_1\text{"}.$$

Let q_0 decide the value of this k to be k_{t_n}. Then

$$A \overset{\text{def}}{=} \{ \zeta : q_1 \text{ does not force } \zeta \notin \underset{\sim}{A}\}$$

is uncountable. Now in both cases we can find an ordinal ζ_n such that: $\zeta_n \geqq \delta_{m+1}$, $\zeta_{t_n} > h(p^n_s)$ for every $s \in {}^{m+1}2$, and there is $q_0 \leqq q_1$ such that $q_1 \Vdash$ "$\zeta_{t_n} \in \underset{\sim}{A}$" and wlog is $q_1 \in D$. Easily $p^{n+1}_{t_n} \overset{\text{df}}{=} q_1$ satisfies the relevant case of (6) and also (5) holds. Now we have to define p^{n+1}_s ($s \in {}^{m+1}2 \setminus \{t_n\}$) to satisfy (6) too; this is easy too.

So we have finished the induction on n, thus we finish the induction step for the induction on m.

Now note that from (9) and the fact that $\sigma(i) \geqq 1$ for $i \in \omega$ it follows that $\{p_t(0); t \in T\}$ is linearly ordered. Hence it is possible to find $r: \delta \times \omega \to \delta$ such that $r \leqq p_t(0)$ for $t \in T$. From (5) it follows that $\{\zeta_t; t \in T\}$ has order type ω and hence there is $r^*: (\delta + 1) \times \omega \to \delta$ such that $\{\zeta_t; t \in T\} = \{r^*(\delta, i); i \in \omega\}$ and $r^* \leqq r$. Note that $r^* \Vdash$ "$\delta \in S$".

Let the order type of Σ be θ and let Σ be enumerated in order as $\{\rho(\gamma); \gamma \in \theta\}$. A descending sequence of conditions $\{q_\gamma \in \mathbf{P}(\rho(\gamma)) \cap p; \gamma < \theta\}$ and a function $J: (\theta \setminus \{0\}) \to 2$ will now be defined by induction on $\gamma < \theta$. It will also be insisted that if $A(\beta) \overset{\mathrm{df}}{=} \{t \in T; t(i) = J(\alpha) \text{ if } \sigma(i) = \rho(\alpha) \text{ and } \alpha < \beta\}$ then:

(12) $q_\gamma \leqq p_t \restriction \rho(\gamma)$ for all $t \in A(\gamma)$.

To start let $q_1(0) \in \mathrm{L}$ be such that $q_1(0) \leqq r^*$ and $q_1(0) \Vdash$ "$\underset{\sim}{c}_1(\delta) = J(1)$". Now suppose that q_β and $J \restriction (\beta \setminus \{0\})$ have been defined. Find $J(\beta)$ and $q^* \in \mathbf{P}(\rho(\beta))$ such that $q^* \leqq q_\beta$ and $q^* \Vdash$ "$\underset{\sim}{c}_{\rho(\beta)}(\delta) = J(\beta)$."

Choose n such that $\rho(\beta) = \sigma(n)$. It will first be shown that

$$\{p_t(\rho(\beta)); t \in A(\beta + 1)\} \text{ is linearly ordered.}$$

To see that this is so suppose that t and s constitute a counterexample. Let $\sigma(j)$ be minimal such that $t(j) \neq s(j)$. It follows from the definition of $A(\beta)$ that $\rho(\beta) \in \sigma(j)$. Since $t \underset{j}{\sim} s$, $p_t \restriction \sigma(j)$ and $p_s \restriction \sigma(j)$ are comparable, and so are $p_t(\rho(\beta))$ and $p_s(\rho(\beta))$.

It is now possible to find $q_{\beta+1} \leqq q^*$ such that $q_{\beta+1}(\rho(\beta)) \leqq p_t(\rho(\beta))$ for all $t \in A(\beta + 1)$". Since $q_{\beta+1}(\rho(\beta))(\delta)$ has not been defined it is possible to insist that $q_{\beta+1}(\rho(\beta))(\delta) = J(\beta)$ and, hence, $q_{\beta+1} \Vdash d_{\rho(\beta)}(\delta) = J(\beta)$. By extending, if necessary, it may be assumed that $q_{\beta+1} \in \underset{\sim}{D}$. At limit stages of the induction it suffices to use the fact that descending sequences from D have lower bounds provided that their heights converge to $\alpha \notin S$. This can be arranged by having (2) fail at α. For the same reason it is possible to find q such that $q \leqq q_\gamma$ for $\gamma \in \theta$. Clearly $q \Vdash$ "$\underset{\sim}{c}_\gamma(\delta) = J(\rho^{-1}(\gamma)) = d_\gamma(\delta)$" for $\gamma \in \Sigma$. Now choose $K \in \omega$ and $r \leqq q$ such that $r \Vdash$ "$\underset{\sim}{f}(\delta) = K$".

Notice that $A(\theta + 1)$ is simply a branch through T. Furthermore, $k_t \leqq K$ for $t \in T$. To see this suppose that $k_t > K$ for some $t \in A(\theta + 1)$. Then either (7a) or (7b) holds. But since $\delta \notin \mathfrak{M}$ it follows by elementarity that (7a) cannot hold. Since (7b) holds, if $K < k_t$ then the minimality of k_t and the elementarity of $\mathfrak{M}$ will be contradicted. It follows that there is some k such that $k_t = k$ for infinitely many $t \in A(\theta + 1)$. But now it follows that $r \Vdash$ "$\underset{\sim}{f}(\zeta_t) = k$" for infinitely many $t \in A(\theta + 1)$ and hence $r \Vdash$ "$|\{n \in \omega; \underset{\sim}{f}(\underset{\sim}{L}(\delta, n)) = k\}| = \aleph_0$".

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